Orthogonal Lie algebra generated by a bilinear form

$V = \mathbb{F}$-vector space, $\dim V = n < \infty$

b bilinear form on V

ΣΗΜΕΙΩΣΗ 3

$$b: \ V \times V \ni (v, w) \longrightarrow b(u, v) \in \mathbb{F}$$

$$b(\alpha u + \beta v, w) = \alpha b(u, w) + \beta b(v, w)$$
$$b(u, \alpha v + \beta w) = \alpha b(u, v) + \beta b(u, w)$$

$\mathfrak{o}(V, b) =$ *Orthogonal Lie algebra* $\equiv$ the set of all $T \in \mathrm{End}\, V$

$$b(u, Tv) + b(Tu, v) = 0$$

$$[T_1, \ T_2] = T_1 \circ T_2 - T_2 \circ T_1$$

T_1 and $T_2 \in \mathfrak{o}(V, b) \ \rightsquigarrow \ [T_1, \ T_2] \in \mathfrak{o}(V, b)$

ΣΗΜΕΙΩΣΗ 3

V γραμμικός χώρος, b διγραμμική απεικόνιση

$$V\times V \ni (u,v) \xrightarrow[\text{bi-linear}]{b} b(u,v)\in\mathbb{C}$$

$$\mathfrak{o}(b,V)\subset gl(V) \text{ τ.ω. } T\in\mathfrak{o}(b,V) \rightsquigarrow b(u,Tv)+b(Tu,V)$$

$\Longrightarrow$ $\mathfrak{o}(b,V)$ άλγεβρα Lie με commutator

$$[T_1,T_2]=T_1\circ T_2 - T_2\circ T_1$$

Απόδειξη: $b(u,[T_1,T_2]v) = b(u, T_1\circ T_2 v) - b(T_1\circ T_2 u, v) =$

$= b(u, T_1(T_2 v)) - b(T_1(T_2 u), v) = -b(T_1 u, T_2 v) + b(T_2 u, T_1 v) =$

επειδή $T_1\in\mathfrak{o}(b,V)$

επειδή $T_2\in\mathfrak{o}(b,V)$

$= b(T_2(T_1 u), v) - b(T_1(T_2) u, v) = -b(T_1\circ T_2 u, v) + b(T_2\circ T_1 u, v) = -b([T_1,T_2]u,v)$

επομένως το $[T_1,T_2]\in\mathfrak{o}(b,V)$.

Το $gl(V)$ είναι Lie άλγεβρα διότι $gl(V)=L(\mathrm{End}\,V) \rightsquigarrow T_1,T_2,T_3\in\mathfrak{o}(T,V)\subset g(V)$

τότε ισχύει η ταυτότητα Jacobi

$\Longrightarrow$ $\boxed{\mathfrak{o}(b,V) \text{ είναι Lie-algebra}}$

Matrix Formulation for bilinear forms

Ξέρουμε από γρ. άλγεβρα αν $\dim V = n$ τότε $\mathrm{End}\, V \longleftrightarrow M_n(V)$

μιγαδικοί πίνακες $n \times n$

$$V \ni a \longleftrightarrow \bar{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \quad \boxed{\langle \bar{a}, \bar{b} \rangle = \sum_{i=1}^{n} a_i b_i = (a_1, a_2, \ldots, a_n) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}}$$

ορισμός $\langle \bar{a}, \bar{b} \rangle$

$$\mathrm{End}(V) \ni M \longleftrightarrow \mathbb{M} = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1n} \\ M_{21} & M_{22} & \cdots & M_{2n} \\ & \cdots & \cdots & \\ & \cdots & \cdots & \\ M_{n1} & M_{n2} & \cdots & M_{nn} \end{pmatrix} \rightsquigarrow \boxed{\langle \mathbb{M}\bar{a}, \bar{b} \rangle = \langle \bar{a}, \mathbb{M}^{\mathrm{tr}} \bar{b} \rangle}$$

ΣΗΜΕΙΩΣΗ 4

bilinear form $b \longleftrightarrow \mathbb{B}$ matrix $n \times n \rightsquigarrow b(x, y) = \langle \bar{x}, \mathbb{B}\bar{y} \rangle$

$M \in O(V, b)$

$b(Mx, My) = b(x, y) \leftrightsquigarrow \mathbb{M}^{\mathrm{tr}}\mathbb{B}\mathbb{M} = \mathbb{B} \leftrightsquigarrow M^{\mathrm{tr}} b M = b$

ΣΗΜΕΙΩΣΗ 5

Συμβολική γραφή

μηδενικός πίνακας

$T \in \mathfrak{o}(V, b)$

$b(Tx, y) + b(x, Ty) = 0 \leftrightsquigarrow \mathbb{T}^{\mathrm{tr}}\mathbb{B} + \mathbb{B}\mathbb{T} = 0_{n \times n} \leftrightsquigarrow T^{\mathrm{tr}} b + bT = 0$

ΑΣΚΗΣΗ 5

ΣΗΜΕΙΩΣΗ 4

$$V = \mathrm{span}\{\overbrace{e_1, e_2, \ldots, e_n}^{\text{Βαση ανεξάρτητων διανυσμάτων}}\} = \mathbb{C}e_1 + \mathbb{C}e_2 + \cdots + \mathbb{C}e_n$$

"Αλγεβρική" γλώσσα — Γλώσσα πινάκων

◇ $V \ni a = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n \Longleftrightarrow \bar{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ $\qquad a \longleftrightarrow \bar{a}$

$$\langle c, a \rangle \overset{\text{def}}{=} \langle \bar{c}, \bar{a} \rangle$$

◇ $\langle c, a \rangle = \sum_{i=1}^{n} c_i a_i \Longleftrightarrow \langle \bar{c}, \bar{a} \rangle = (c_1, c_2, \ldots c_n) \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \bar{c}^{Tr} \cdot \bar{a}$

$\langle e_i, e_j \rangle = \delta_{ij}$

◇ $\mathrm{End}\,V \ni M : V \xrightarrow[\text{linear}]{} V$ $\qquad \langle \bar{e}_i, \bar{e}_j \rangle = \delta_{ij}$

$\langle e_i, M e_j \rangle = M_{ij} \Longleftrightarrow \mathbb{M} = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1n} \\ M_{21} & M_{22} & \cdots & M_{2n} \\ \vdots & & & \vdots \\ M_{n1} & M_{n2} & \cdots & M_{nn} \end{pmatrix}$ $\qquad \underset{\text{linear map}}{M} \longleftrightarrow \underset{\text{matrix}}{\mathbb{M}}$

◇ $\langle Mc, a \rangle = (\mathbb{M} \cdot \bar{c})^{Tr} \cdot \bar{a} = \bar{c}^{Tr} \cdot \mathbb{M}^{tr} \cdot \bar{a} = \langle \bar{c}, \mathbb{M}^{Tr} \bar{a} \rangle = \langle c, M^{tr} a \rangle$

$\langle c, Ma \rangle = \langle \bar{c}, \mathbb{M} \bar{a} \rangle = \langle \mathbb{M}^{tr} \bar{c}, \bar{a} \rangle = \langle M^{Tr} c, a \rangle$

◇ b bilinear form $\longleftrightarrow$ $\mathbb{B}$ matrix $\Longrightarrow$ $\boxed{b(c,a) = \bar{c}^{tr} \mathbb{B} \bar{a} = \langle \bar{c}, \mathbb{B} \bar{a} \rangle}$

$b(e_i, e_j) = \mathbb{B}_{ij}$

$\boxed{\text{ΣΗΜΕΙΩΣΗ 5}}$ $\quad O(b,V) \ni M$

$b(Mc, Ma) = b(c,a) \rightsquigarrow \langle \bar{c}, \mathbb{M}^{\tau z}\, \mathbb{B}\, \mathbb{M}\bar{a} \rangle = \langle \bar{c}, \mathbb{B}\bar{a} \rangle \rightsquigarrow \mathbb{M}^{tz}\, \mathbb{B}\, \mathbb{M} = \mathbb{B}$

Απόδειξη: $b(Mc, Ma) = (\mathbb{M}\bar{c})^{tz} \cdot \mathbb{B} \cdot (\mathbb{M}\bar{a}) = \bar{c}^{tz} \cdot \mathbb{M}^{tz} \cdot \mathbb{B} \cdot \mathbb{M}\bar{a} = \bar{c}^{tz} \cdot \mathbb{B} \cdot \bar{a}$

Lie algebra of the (Lie) group of isometries

V $\mathbb{F}$-vector space, $b : V \otimes V \longrightarrow \mathbb{C}$ bilinear form

$O(V, b)$=group of isometries on V i.e.

$$O(V, b) \ni M : V \underset{\text{lin}}{\longrightarrow} V \rightsquigarrow b(Mu, Mv) = b(u, v)$$

$$\left.\begin{array}{r} M(t) \text{ smooth trajectory } \in O(V, b) \\ M(0) = I \end{array}\right\} \rightsquigarrow$$

$\boxed{\text{ΣΗΜΕΙΩΣΗ 6}}$

smooth trajectory
$M(t) \in M_n(\mathbb{C})$.
$M_{ij}(t)$ smooth function
i.e. $M_{ij}(t) \in C^\infty([0,1])$, $t \in [0,1]$
δηλ. $M_{ij}(t)$ έχει παραγώγους κάθε τάξης συνεχείς.

$$M'(0) = T \in \mathfrak{o}(V, b) = \text{ Lie algebra}$$

$$\Longrightarrow b(Tu, v) + b(u, Tv) = 0$$

$$\Longrightarrow T^{\mathrm{tr}} b + b\,T = 0$$

εφαπτόμενος χώρος στην $\mathbb{I}$

group of isometries $O(V, b) \longrightarrow$ Lie algebra $\mathfrak{o}(V, b) = T_{\mathbb{I}}(O(V, b))$

The Lie algebra $\mathfrak{o}(V, b)$ is the tangent space of the manifold $O(V, b)$ at the unity $\mathbb{I}$

$\boxed{\text{ΣΗΜΕΙΩΣΗ 6}}$

$$\boxed{\begin{array}{l} M(t) \in O(b,V) \\ \text{και} \quad M(0) = Id \end{array} \Rightarrow \left.\frac{dM(t)}{dt}\right|_{t=0} = T \in o(b,V)}$$

Απόδειξη:

$$b(M(t)c, M(t)a) = \bar{c}^{T_z} \cdot M^{t_z}(t)\, \mathbb{B}\, M(t)\bar{a} = \bar{c}^{T_z} \cdot \mathbb{B} \cdot \bar{a} \Rightarrow \frac{d}{dt}\langle M(t)c, M(t)a\rangle = 0$$

$$\frac{d}{dt}\langle M(t)c, M(t)a\rangle = \bar{c}^{T_z} \cdot \frac{dM^{t_z}(t)}{dt}\mathbb{B} \cdot M(t)\bar{a} + \bar{c}^{T_z} M^{t_z}(t) \cdot \mathbb{B}\frac{dM(t)}{dt} \cdot \bar{a} = 0$$

Για $t=0 \rightsquigarrow \left.\frac{dM}{dt}\right|_{t=0} = T, \left.\frac{dM^{T_z}}{dt}\right|_{t=0} = T^{T_z}, M(0) = I, M^{t_z}(0) = I \Rightarrow \bar{c}^{T_z} \cdot T^{T_z} \cdot \mathbb{B} \cdot I\bar{a} + \bar{c}^{T_z} I \cdot \mathbb{B} \cdot T\bar{a} = 0$

$$\langle \bar{c}, (T^{t_z} \cdot \mathbb{B} + \mathbb{B} \cdot T)\bar{a}\rangle = 0 \rightsquigarrow T^{T_z} \cdot \mathbb{B} + \mathbb{B} \cdot T = 0 \Rightarrow \boxed{T \in o(b,V)}$$

ΓΕΝΙΚΟ ΣΥΜΠΕΡΑΣΜΑ

$$\boxed{\text{ΙΣΟΜΕΤΡΙΑ} \Rightarrow \text{Lie ΑΛΓΕΒΡΑ}}$$

Rotation Group $SO(3)$ and Lie Algebra $\mathfrak{so}(3)$

$$V = \mathbb{R}^3 \text{ and } b(\overline{a}, \overline{b}) = <\overline{a}, \overline{b}> = \sum_{i=1}^{3} a_i b_i \implies \mathbb{B} = \mathbb{I}$$

Θα μελετηθεί αναλυτικά στη Διαφορική Γεωμετρία

Isometries on V are the orthogonal matrices $\mathbb{M}^{\mathrm{Tr}}\mathbb{M} = \mathbb{I}$

The Group of Rotations $SO(3) = O(V, b)$

Rotations on V are the orthogonal matrices $\mathbb{M}^{\mathrm{Tr}}\mathbb{M} = \mathbb{I}$ with $\mathrm{Det}\,\mathbb{M} = 1$

$$M(\phi, \psi, \theta) =$$
$$= \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

Exercise

Find the Lie Algebra $\mathfrak{o}(V, b) = \mathfrak{so}(3)$ commutation relations

$\mathfrak{gl}(V)$ algebra

$V = \mathbb{F}$-vector space, $\dim V = n < \infty$

general linear algebra $\mathfrak{gl}(V)$

$\mathfrak{gl}(V) \equiv \mathrm{End}(V) =$ endomorphisms on V $\dim \mathrm{gl}(V) = n^2$
$\mathfrak{gl}(V)$ is a Lie algebra with commutator $[A, B] = AB - BA$

$\mathfrak{gl}(\mathbb{C}^n) \equiv \mathfrak{gl}(n, \mathbb{C}) \equiv \mathbf{M}_n(\mathbb{C}) \equiv$ complex $n \times n$ matrices

Standard basis

Standard basis for $\mathfrak{gl}(n, \mathbb{C}) =$ matrices E_{ij} (1 in the (i, j) position, 0 elsewhere) $[E_{ij}, E_{k\ell}] = \delta_{jk} E_{i\ell} - \delta_{i\ell} E_{kj}$

Jordan-Chevalley decomposition

- $A \in gl(\mathbb{C}^n) \rightsquigarrow$ we can find $W \in GL(\mathbb{C}^n)$ such that $WAW^{-1} = A_s + A_n$, $[A_s, A_n] = A_s A_n - A_n A_s = 0$
- A_s diagonal matrix (or semisimple)
- A_n nilpotent matrix $(A_n)^m = 0$, the decomposition is unique

(Humphreys 1972, Prop. 4.2, p. 17)

ΒΑΣΙΚΟ ΘΕΩΡΗΜΑ ΓΡΑΜΜΙΚΗΣ ΑΛΓΕΒΡΑΣ

$x = x_s + x_n$, x_s semisimple = diagonalizable matrix, x_n nilpotent matrix

πίνακας στο $M_n(\mathbb{C})$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$e^x = e^{(x_s + x_n)} = e^{x_s} e^{x_n}$$

Επειδή $x_s x_n = x_n x_s$

$$x_s \text{ diagonal matrix } = \begin{pmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & \lambda_n \end{pmatrix} \Rightarrow$$

$$x_s^m = \begin{pmatrix} \lambda_1^m & & 0 \\ & \ddots & \\ 0 & & \lambda_n^m \end{pmatrix}$$

$$e_s^x \text{ diagonal matrix } = \begin{pmatrix} e^{\lambda_1} & 0 & 0 & \dots & 0 \\ 0 & e^{\lambda_2} & 0 & \dots & 0 \\ & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & e^{\lambda_n} \end{pmatrix}$$

$$e^{x_n} = \sum_{k=0}^{m-1} \frac{x_n^k}{k!}$$

ΣΥΜΠΕΡΑΣΜΑ

$\boxed{\text{Για κάθε } x \in M_n(\mathbb{C}) \text{ υπάρχει το } e^x}$

Corrolary of the Chinese Theorem

There are polynomials $P(x)$ and $Q(x)$ such that $x_s = P(x)$ and $x_n = Q(x)$

General Linear Group $GL(n, \mathbb{F})$

$\mathrm{GL}(n, \mathbb{F})$ =invertible $n \times n$ matrices
$M \in \mathrm{GL}(n, \mathbb{F}) \leftrightsquigarrow \det M \neq 0$

$\mathrm{GL}(n, \mathbb{C})$ is an analytic manifold of dim n^2 AND a group.

The group multiplication is a continuous application $G \times G \longrightarrow G$
The group inversion is a continuous application $G \longrightarrow G$

Definition of the Lie algebra from the Lie group

$$\mathfrak{g}(n, \mathbb{C}) = T_{\mathbb{I}}\left(\mathrm{GL}(n, \mathbb{C})\right)$$

$$\left.\begin{array}{c} X(t) \in \mathrm{GL}(n, \mathbb{C}) \\ X(t) \text{ smooth trajectory} \\ X(0) = \mathbb{I} \end{array}\right\} \Longrightarrow \left\{X'(0) = x \in \mathfrak{gl}(n, \mathbb{C})\right\}$$

Definition of a Lie group from a Lie algebra

$$\{x \in \mathfrak{gl}(n, \mathbb{C})\} \Longrightarrow \left\{X(t) = \mathrm{e}^{xt} = \sum_{n=0}^{\infty} \frac{t^n}{n!} x^n \in \mathrm{GL}(n, \mathbb{C}^{\times})\right\}$$

Structure Constants

$$\mathfrak{g} = \mathrm{span}\left(e_1,\, e_2,\, \ldots,\, e_n\right) = \mathbb{F}e_1 + \mathbb{F}e_2 + \cdots + \mathbb{F}e_n$$

$$[e_i,\, e_j] = \sum_{k=1}^{n} c_{ij}^{k}\, e_k \equiv c_{ij}^{\mathbf{k}}\, e_{\mathbf{k}},\; c_{ij}^{k} \longleftrightarrow \textbf{structure constants}$$

(i) **bi-linearity**

(ii) **anti-commutativity**: $c_{ij}^{k} = -c_{ji}^{k}$

(iii) **Jacobi identity**: $c_{i\,j}^{\mathbf{m}} c_{\mathbf{m}\,k}^{\ell} + c_{j\,k}^{\mathbf{m}} c_{\mathbf{m}\,i}^{\ell} + c_{k\,i}^{\mathbf{m}} c_{\mathbf{m}\,j}^{\ell} = 0$

$\longrightarrow$ summation on **color** indices

$\boxed{\text{ΑΣΚΗΣΗ 6}}$

Classical Lie Algebras

$\boxed{\mathtt{A}_\ell}$ or $\mathfrak{sl}(\ell+1, \mathbb{C})$ or **s**pecial **l**inear algebra

Def: $(\ell+1) \times (\ell+1)$ complex matrices T with $\mathrm{Tr}\, T = 0$

Dimension $= \dim(\mathtt{A}_\ell) = (\ell+1)^2 - 1 = \ell(\ell+2)$

Standard Basis: E_{ij}, $i \neq j$, $i, j = 1, 2 \ldots, \ell+1$. $H_i = E_{ii} - E_{i+1\, i+1}$

$1 \leq i \leq \ell$

$\boxed{\text{ΑΣΚΗΣΗ 7}}$

Να βρεθούν οι σχέσεις μετάθεσης για την A_1 και την A_2 (δύσκολο, προαιρετικό)

$\boxed{C_\ell}$ or $\mathfrak{sp}(2\ell, \mathbb{C})$ or **sym**ple**c**tic algebra

Def: $(2\ell) \times (2\ell)$ complex matrices T with $\mathrm{Tr}\, T = 0$ leaving infinitesimally invariant the bilinear form

$$b \longleftrightarrow \begin{pmatrix} 0_\ell & \mathbb{I}_\ell \\ -\mathbb{I}_\ell & 0_\ell \end{pmatrix}$$

$$\left.\begin{array}{c} U,\, V \in \mathbb{C}^{2\ell} \\ T \in \mathfrak{sp}(2\ell, \mathbb{C}) \end{array}\right\} \rightsquigarrow b(U, TV) + b(TU, V) = 0$$

$$T = \begin{pmatrix} \mathbb{M} & \mathbb{N} \\ \mathbb{P} & -\mathbb{M}^{\mathrm{tr}} \end{pmatrix} \rightsquigarrow \left\{ \begin{array}{c} \mathbb{N}^{\mathrm{tr}} = \mathbb{N}, \\ \mathbb{P}^{\mathrm{tr}} = \mathbb{P} \end{array} \right.$$

$$bT + T^{\mathrm{tr}} b = 0 \;\rightsquigarrow\; bTb^{-1} = -T^{\mathrm{tr}}$$

ΑΣΚΗΣΗ 8

$\boxed{\mathrm{B}_\ell}$ or $\mathfrak{o}(2\ell+1,\mathbb{C})$ or (odd) **o**rthogonal algebra

Def: $(2\ell+1)\times(2\ell+1)$ complex matrices T with $\mathrm{Tr}\, T = 0$ leaving infinitesimally invariant the bilinear form

$$b \longleftrightarrow \begin{pmatrix} 1 & \overline{0} & \overline{0} \\ \overline{0}^{\mathrm{tr}} & 0_\ell & \mathbb{I}_\ell \\ \overline{0}^{\mathrm{tr}} & \mathbb{I}_\ell & 0_\ell \end{pmatrix}$$

$\overline{0} = (0, 0, \ldots, 0)$

$$\left.\begin{array}{c} U,\ V \in \mathbb{C}^{2\ell} \\ T \in \mathfrak{o}(2\ell+1,\mathbb{C}) \end{array}\right\} \rightsquigarrow b(U, TV) + b(TU, V) = 0$$

$$T = \begin{pmatrix} 0 & \overline{b} & \overline{c} \\ -\overline{c}^{\mathrm{tr}} & \mathbb{M} & \mathbb{N} \\ -\overline{b}^{\mathrm{tr}} & \mathbb{P} & -\mathbb{M}^{\mathrm{tr}} \end{pmatrix} \rightsquigarrow \left\{ \begin{array}{c} \mathbb{N}^{\mathrm{tr}} = -\mathbb{N}, \\ \mathbb{P}^{\mathrm{tr}} = -\mathbb{P} \end{array} \right.$$

$\boxed{\mathsf{D}_\ell}$ or $\mathfrak{o}(2\ell, \mathbb{C})$ or (even) **o**rthogonal algebra

Def: $(2\ell) \times (2\ell)$ complex matrices T with $\mathrm{Tr}\, T = 0$ leaving infinitesimally invariant the bilinear form

$$b \longleftrightarrow \begin{pmatrix} 0_\ell & \mathbb{I}_\ell \\ \mathbb{I}_\ell & 0_\ell \end{pmatrix}$$

$$\left. \begin{array}{c} U, V \in \mathbb{C}^{2\ell} \\ T \in \mathfrak{o}(2\ell, \mathbb{C}) \end{array} \right\} \rightsquigarrow b(U, TV) + b(TU, V) = 0$$

$$T = \begin{pmatrix} \mathbb{M} & \mathbb{N} \\ \mathbb{P} & -\mathbb{M}^{\mathrm{tr}} \end{pmatrix} \rightsquigarrow \left\{ \begin{array}{c} \mathbb{N}^{\mathrm{tr}} = -\mathbb{N}, \\ \mathbb{P}^{\mathrm{tr}} = -\mathbb{P} \end{array} \right.$$